

HIGHER DIMENSIONAL SCHWARZ'S SURFACES AND SCHERK'S SURFACES

JAIGYOUNG CHOE AND JENS HOPPE

ABSTRACT. Higher dimensional generalizations of Schwarz's P -surface, Schwarz's D -surface and Scherk's second surface are constructed as complete embedded periodic minimal hypersurfaces in \mathbb{R}^n .

In \mathbb{R}^3 minimal surfaces are easy to construct. Thanks to the existence of isothermal coordinates on a surface, one can derive the Weierstrass representation formula, which allows one to obtain minimal surfaces in \mathbb{R}^3 at will. Nonetheless, only a few topologically simple complete minimal surfaces were known to exist in \mathbb{R}^3 until recently.

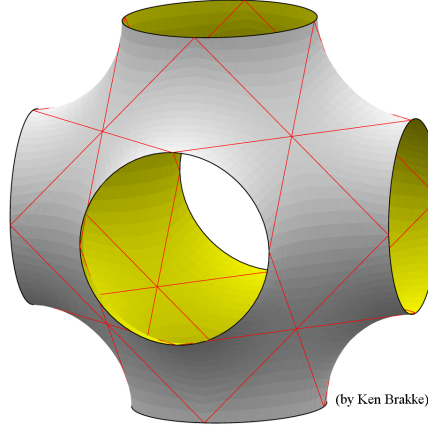
It is not easy to understand the topology of a minimal surface in terms of its Weierstrass data. It is ironical that many of these well-known simple minimal surfaces could be constructed without resorting to the Weierstrass representation formula. The catenoid, the helicoid, Enneper's surface, Scherk's first surface, Scherk's second surface, Schwarz's P -surface and Schwarz's D -surface can be constructed by exploiting their geometric characteristics.

In $\mathbb{R}^n, n \geq 4$, there is no systematic method to construct minimal hypersurfaces. So far, only the catenoid [B], the helicoid [CH] and Enneper's surface [C] are known to have higher dimensional versions in \mathbb{R}^n . In this paper we construct the higher dimensional generalizations of Schwarz's P -surface, Schwarz's D -surface and Scherk's second surface. First, we extract geometric characteristics of their fundamental pieces, and then solve the Dirichlet problem to construct the higher dimensional versions of the fundamental pieces and extend them across their boundaries by 180° -rotation.

1. SCHWARZ'S P -SURFACE

A triply periodic minimal surface in \mathbb{R}^3 was first constructed by H.A. Schwarz [S] in 1865. It was found as a by-product in the process of solving the Plateau problem in a concrete case. The Jordan curve that Schwarz considered was the skew quadrilateral Γ consisting of the four edges of a regular tetrahedron T . He found the minimal surface S_0 spanning Γ from explicit data for the Weierstrass representation formula. Since T fits nicely in a cube Q^3 so that each edge of Γ becomes a diagonal on the square faces of Q^3 , Schwarz was able to show that the analytic extension S of S_0 is an embedded triply periodic minimal surface in \mathbb{R}^3 . He also proved that S^* , the conjugate minimal surface of S , is embedded and triply periodic as well. The quadrilateral Γ^* bounding the fundamental piece of S^* has vertex angles of $\pi/3, \pi/3, \pi/2, \pi/2$ while those of Γ are $\pi/3, \pi/3, \pi/3, \pi/3$. Because of the vertex angles $\pi/2, \pi/2$ of Γ^* S^* turns out to be perpendicular to ∂Q^3 . Moreover, due to the vertex angles $\pi/3, \pi/3$ of Γ^* as well as Γ , both S and S^* contain three straight lines meeting at every flat point. In fact S^* is the well-known Schwarz P -surface. Figure 1 shows a fundamental piece of S^* in $[-3, 1] \times [-3, 1] \times [-3, 1]$:

J.C. supported in part by NRF 2011-0030044, SRC-GAIA.



S^* : Schwarz P -surface

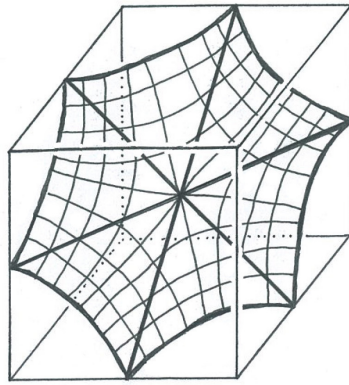
Figure 1

Part of S^* in a smaller cube Q^3 is shown in Figure 2. This part, denoted H , is diffeomorphic to a hexagon and consists of 6 congruent triangular pieces. Each triangular piece is bounded by two line segments and a planar curve. Along this curve the triangular piece is perpendicular to the face of the cube. H is close to the regular hexagon H_0 such that $L := H \cap H_0$ is three straight lines meeting each other at 60° . Let's introduce a coordinate system (x_1, x_2, x_3) such that

$$Q^3 = [-1, 1] \times [-1, 1] \times [-1, 1] \quad \text{and} \quad H_0 = \{(x_1, x_2, x_3) \in Q^3 : x_1 + x_2 + x_3 = 0\}.$$

Then the three straight lines L in H are the intersection of H_0 with the three coordinate planes of \mathbb{R}^3 and furthermore

$$L = H_0 \cap \{(x_1, x_2, x_3) : x_1 + x_2 = 0 \text{ or } x_2 + x_3 = 0 \text{ or } x_1 + x_3 = 0\}.$$



$$H = S^* \cap Q^3$$

Figure 2

We will generalize these properties of $S^* \cap Q^3$ to find a higher dimensional Schwarz surface in \mathbb{R}^n . First let Q^n be the n -dimensional cube in \mathbb{R}^n

$$Q^n = [-1, 1]^n = \{(x_1, \dots, x_n) : -1 \leq x_i \leq 1\}.$$

Define

$$P_n = \{(x_1, \dots, x_n) \in Q^n : x_1 + \dots + x_n = 0\}.$$

Then P_n is an $(n-1)$ -dimensional polyhedron with $2n$ faces, that is,

$$\partial P_n = \left(\bigcup_{i=1}^n B_i^+ \right) \cup \left(\bigcup_{i=1}^n B_i^- \right),$$

where

$$B_i^+ = \{(x_1, \dots, x_n) \in \partial Q^n : x_i = 1, x_1 + \dots + \widehat{x_i} + \dots + x_n = -1\},$$

$$B_i^- = \{(x_1, \dots, x_n) \in \partial Q^n : x_i = -1, x_1 + \dots + \widehat{x_i} + \dots + x_n = 1\}.$$

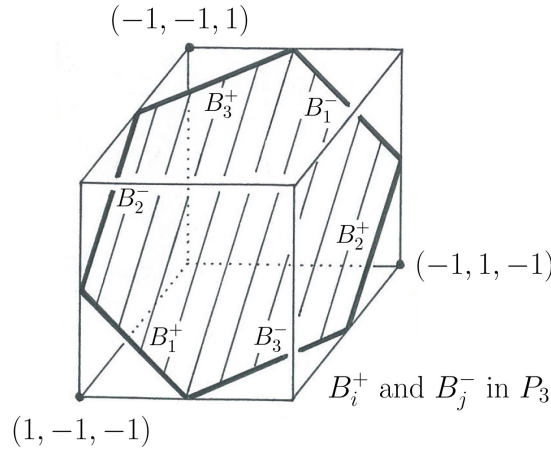


Figure 3

P_3 is a regular hexagon and P_4 is a regular octahedron. What can one say about P_n ? Let $G_1 \subset O(n)$ be the group of all isometries of \mathbb{R}^n which act on $\{x_1, \dots, x_n\}$ as permutations and define $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $\varphi(x) = -x$, $x \in \mathbb{R}^n$. Let G_2 be the subgroup of $O(n)$ generated by $G_1 \cup \{\varphi\}$. Then for any B_i^+ and B_j^- there exist isometries $\psi_1 \in G_1$ and $\psi_2 \in G_2$ such that

$$\psi_1(B_1^+) = B_i^+ \quad \text{and} \quad \psi_2(B_1^+) = B_j^-.$$

Therefore one can say that the faces of P_n are congruent to each other.

More precisely,

$$\begin{aligned} B_n^+ &= \{x_n = 1, x_1 + \dots + x_{n-1} = -1\} \cap Q^n \\ &= \{x_n = 1, x_1 + \dots + x_{n-1} = -1\} \cap \{-1 \leq x_1, \dots, x_{n-1}\} \cap \{x_1, \dots, x_{n-1} \leq 1\} \\ &= \{x_n = 1, x_1 + \dots + x_{n-1} = -1\} \cap \{-1 \leq x_1, \dots, x_{n-1} \leq n-3\} \cap \{x_1, \dots, x_{n-1} \leq 1\} \\ &= A \cap \{x_1, \dots, x_{n-1} \leq 1\}, \end{aligned}$$

where $A := \{x_n = 1, x_1 + \dots + x_{n-1} = -1\} \cap \{-1 \leq x_1, \dots, x_{n-1} \leq n-3\}$ is the regular $(n-2)$ -simplex with vertices $(n-3, -1, \dots, -1, 1)$, $(-1, n-3, -1, \dots, -1, 1)$, \dots , $(-1, \dots, -1, n-3, 1)$. Then B_n^+ is the truncated regular $(n-2)$ -simplex, i.e., truncated by the half spaces $\{1 < x_i\}$, $i = 1, \dots, n-1$, at all its vertices. In case $n = 3$ and 4 , B_i^\pm is the regular $(n-2)$ -simplex with no truncation.

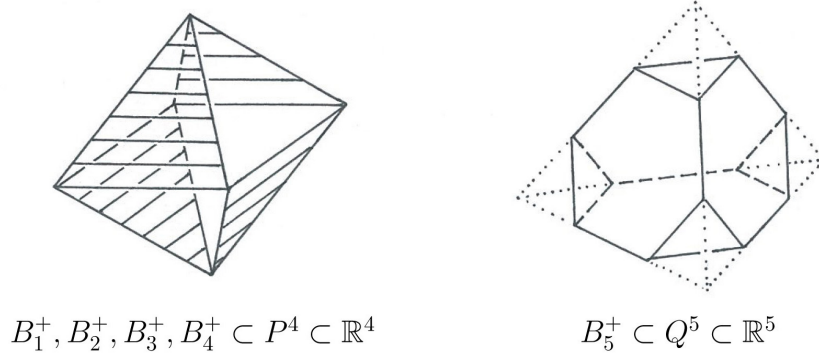


Figure 4

In dimension $n \geq 5$, the faces of B_i^\pm consist of the faces of the regular $(n-2)$ -simplex and those created by the truncation. In other words,

$$\partial B_i^\pm = \left(\bigcup_{j=1}^{n-1} F_j \right) \cup \left(\bigcup_{j=1}^{n-1} \hat{F}_j \right),$$

where F_j is a subset of a face of the $(n-2)$ -simplex and \hat{F}_j is the face created by the truncation at each vertex. In dimension $n = 3, 4$, however, $\partial B_i^\pm = \bigcup_{j=1}^{n-1} F_j$.

The three straight lines $L = H \cap H_0$ mentioned above is called the *spine* of H (or of H_0). The spine L_n of P_n is defined as

$$L_n = \left(\bigcup_{i=1}^n O \rtimes \partial B_i^+ \right) \cup \left(\bigcup_{i=1}^n O \rtimes \partial B_i^- \right),$$

where O is the origin of \mathbb{R}^n and $O \rtimes \partial B_i^\pm$ denotes the cone which is the union of all the line segments from O over ∂B_i^\pm . In fact

$$L_n = O \rtimes (P_n \cap (n-2)\text{-skeleton of } Q^n).$$

Since $F_j \subset \partial A$ on ∂B_n^\pm , we have

$$\bigcup_{j=1}^{n-1} F_j \subset Q^n \cap \bigcup_{j=1}^{n-1} \{x_j = \mp 1, x_n = \pm 1, x_1 + \cdots + \hat{x}_j + \cdots + x_{n-1} = 0\},$$

and hence

$$\begin{aligned} (O \rtimes \partial B_n^+) \cup (O \rtimes \partial B_n^-) &\supset O \rtimes \bigcup_{j=1}^{n-1} F_j \\ &\subset \left[\bigcup_{j=1}^{n-1} \{x_j + x_n = 0\} \right] \cap \{x_1 + \cdots + x_n = 0\} \cap Q^n. \end{aligned}$$

Therefore for $n = 3, 4$, $\hat{F}_j = \emptyset$ and we have

$$L_n = \left[\bigcup_{1 \leq i \neq j \leq n} \{x_i + x_j = 0\} \right] \cap P_n.$$

Actually, $L_3 = L$ is the three straight lines on $P_3 = H_0$ and L_4 is the three mutually orthogonal 2-planes on P_4 :

$$L_4 = (\{x_1 + x_2 = 0\} \cup \{x_1 + x_3 = 0\} \cup \{x_1 + x_4 = 0\}) \cap P_4.$$

For $n \geq 5$, however, because of the nonempty set $\cup_j \hat{F}_j$ we can just say that

$$\mathcal{H}^{n-2} \left(L_n \cap \left[\bigcup_{1 \leq i \neq j \leq n} \{x_i + x_j = 0\} \right] \cap P_n \right) > 0,$$

where \mathcal{H}^{n-2} denotes the $(n-2)$ -dimensional Hausdorff measure.

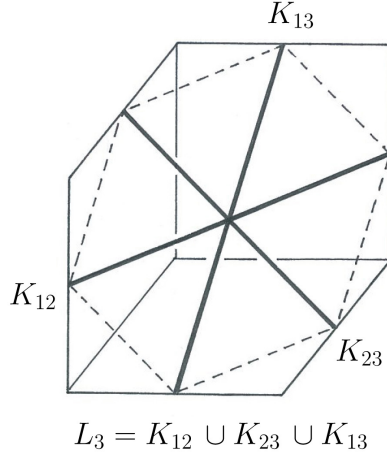


Figure 5

Now fixing the spine L_n of P_n , we want to perturb P_n into a minimal hypersurface Σ_4 in Q^n . First, for an $(n-2)$ -plane K in \mathbb{R}^n , we need to define the 180°-rotation ρ_K of \mathbb{R}^n around K . Let

$$K_{12} = \{x_1 + x_2 = 0\} \cap \{x_1 + \cdots + x_n = 0\}.$$

Then both $u := (1, 1, 0, \dots, 0)$ and $v := (0, 0, 1, \dots, 1)$ are orthogonal to K_{12} . Hence the foot of perpendicular from (x_1, \dots, x_n) to K_{12} is

$$(x_1, \dots, x_n) - \frac{x_1 + x_2}{2} u - \frac{x_3 + \cdots + x_n}{n-2} v.$$

Since the foot of perpendicular is the midpoint of $\mathbf{x} := (x_1, \dots, x_n)$ and $\rho_{K_{12}}(\mathbf{x})$, we have

$$(1.1) \quad \rho_{K_{12}}(\mathbf{x}) = \left(-x_2, -x_1, x_3 - \frac{2}{n-2}(x_3 + \cdots + x_n), \dots, x_n - \frac{2}{n-2}(x_3 + \cdots + x_n) \right).$$

In general, if we define

$$K_{ij} = \{x_i + x_j = 0\} \cap \{x_1 + \cdots + x_n = 0\},$$

the i th and j th components of $\rho_{K_{ij}}(x_1, \dots, x_n)$ are $-x_j$ and $-x_i$, respectively.

Note that for all n ,

$$\rho_{K_{ij}}(\tilde{P}_n) = \tilde{P}_n, \text{ if } \tilde{P}_n := \{x_1 + \cdots + x_n = 0\}.$$

For $n = 3, 4$, we see that

$$(1.2) \quad \rho_{K_{ij}}(L_n) = L_n, \quad \rho_{K_{ij}}(Q^n) = Q^n \quad \text{and} \quad \rho_{K_{ij}}(P_n) = P_n$$

because

$$\rho_{K_{12}}(x_1, x_2, x_3) = (-x_2, -x_1, -x_3),$$

and

$$\rho_{K_{12}}(x_1, x_2, x_3, x_4) = (-x_2, -x_1, -x_4, -x_3).$$

Unfortunately, however, for $n \geq 5$ we have

$$(1.3) \quad \rho_{K_{ij}}(L_n) \neq L_n, \quad \rho_{K_{ij}}(Q^n) \neq Q^n \quad \text{and} \quad \rho_{K_{ij}}(P_n) \neq P_n,$$

because

$$\rho_{K_{ij}}(\{x_k = 1\}) \neq \{x_l = -1\} \quad \text{for any } l \text{ if } k \neq i, j,$$

even though

$$\rho_{K_{ij}}(\{x_i = 1\}) = \{x_j = -1\}.$$

Let $\hat{O} = (2, 0, \dots, 0)$ and consider $(O \rtimes B_1^+) \cup (\hat{O} \rtimes B_1^+)$ and

$$\Gamma_1 := (O \rtimes \partial B_1^+) \cup (\hat{O} \rtimes \partial B_1^+).$$

Here we want to deform $(O \rtimes B_1^+) \cup (\hat{O} \rtimes B_1^+)$ into a minimal hypersurface spanning Γ_1 . Let Π_1 be the orthogonal projection of \mathbb{R}^n onto the hyperplane $\{x_2 + \dots + x_n = 0\}$. Note that $\Pi_1(O \rtimes B_1^+ \cup \hat{O} \rtimes B_1^+)$ contains $\overline{O\hat{O}}$. This fact, together with the convexity of B_1^+ in $\{x_1 = 1\}$, implies that $\Pi_1(O \rtimes B_1^+ \cup \hat{O} \rtimes B_1^+)$ is convex on $\{x_2 + \dots + x_n = 0\}$. Since Γ_1 is the graph of a piecewise linear function on $\Pi_1(\Gamma_1)$, Jenkins-Serrin's theorem [JS] states that there exists a unique minimal hypersurface Σ_0 spanning Γ_1 as a graph over $\Pi_1(O \rtimes B_1^+ \cup \hat{O} \rtimes B_1^+)$. (See Figure 6). Let $\Sigma_1 = \Sigma_0 \cap Q^n$. From the symmetry of Γ_1 with respect to $\{x_1 = 1\}$ it follows that Σ_1 is also symmetric with respect to $\{x_1 = 1\}$ and hence Σ_1 is perpendicular to $\{x_1 = 1\}$ along its boundary on $\{x_1 = 1\}$.

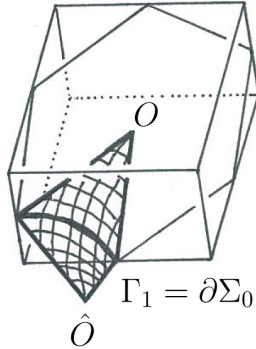


Figure 6

Recall that G_1 is the subgroup of $O(n)$ consisting of all the isometries acting on $\{x_1, \dots, x_n\}$ as permutations. Let G_0 be the subgroup of G_1 consisting of all the permutations of $\{x_1, \dots, x_n\}$ fixing x_1 . Note that Γ_1 is invariant under any $\psi \in G_0$. Hence the uniqueness of the minimal graph Σ_0 spanning Γ_1 implies that Σ_0 is also invariant under G_0 .

We now try to extend Σ_1 analytically to obtain a complete minimal hypersurface in \mathbb{R}^n as follows. Define

$$\Sigma_2 = \bigcup_{\psi \in G_1} \psi(\Sigma_1), \quad \Sigma_3 = \bigcup_{\psi \in G_1} \psi(\varphi(\Sigma_1)), \quad \Sigma_4 = \Sigma_2 \cup \Sigma_3,$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\varphi(x) = -x$. (See Figure 7.) Clearly $\psi(L_n) = L_n$ for any $\psi \in G_2$. From the invariance of Σ_0 under G_0 we see that if $\psi_1(\Sigma_1)$ and $\psi_2(\Sigma_1)$, $\psi_1, \psi_2 \in G_1$, span the same

boundary inside Q^n , i.e., if $\psi_1(\Sigma_1) \setminus \partial Q^n = \psi_2(\Sigma_1) \setminus \partial Q^n$, then they must coincide. Hence both Σ_2 and Σ_3 are embedded. Moreover, we have

$$\partial\Sigma_2 \cap \partial\Sigma_3 = L_n, \quad \partial\Sigma_2 \setminus L_n \subset \partial Q^n, \quad \partial\Sigma_3 \setminus L_n \subset \partial Q^n.$$

Hence Σ_4 is a connected, C^0 , piecewise analytic manifold with $\partial\Sigma_4 \subset \partial Q^n$.

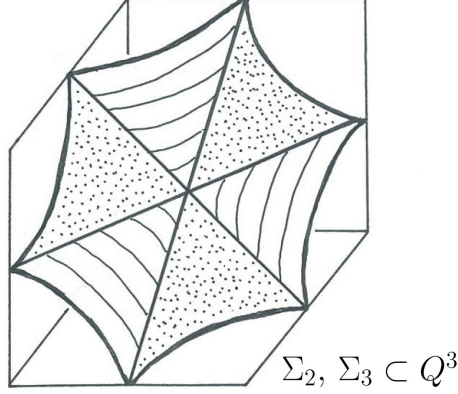


Figure 7

We claim here that Σ_4 is an analytic extension of Σ_1 only when $n = 3, 4$. From the well-known removable singularity theorem (Theorem 1.4, [HL]) it follows that the following four statements are equivalent:

- Σ_4 is an analytic extension of Σ_1 .
- \Leftrightarrow The tangent planes to Σ_1 and to $\Sigma_4 \setminus \Sigma_1$ coincide at every point of $\Sigma_1 \cap \Sigma_4 \cap K_{12}$.
- $\Leftrightarrow \rho_{K_{12}}(\Sigma_1)$ is a subset of Σ_4 .
- \Leftrightarrow

$$(1.4) \quad \rho_{K_{12}}(O \rtimes \partial B_1^+) = O \rtimes \partial B_2^-.$$

Remark that

$$(O \rtimes \cup_j F_j) \cap (O \rtimes \partial B_1^+) \subset \cup_{i \neq 1} \{x_1 + x_i = 0\} \cap \{x_1 + \cdots + x_n = 0\} \cap Q^n$$

and

$$(1.5) \quad (O \rtimes \cup_j F_j) \cap (O \rtimes \partial B_2^-) \subset \cup_{i \neq 2} \{x_2 + x_i = 0\} \cap \{x_1 + \cdots + x_n = 0\} \cap Q^n.$$

From (1.1) we see that the sum of the second and k -th components of $\rho_{K_{12}}(x_1, \dots, x_n)$ for $(x_1, \dots, x_n) \in (O \rtimes \cup_j F_j) \cap (O \rtimes \partial B_1^+)$ equals

$$-x_1 + x_k - \frac{2}{n-2}(x_3 + \cdots + x_n),$$

which does not vanish when $x_1 + x_i = 0$, $i \neq 1$, and $x_1 + \cdots + x_n = 0$, if $k \geq 3$ and $n \geq 5$. It follows from (1.5) that $O \rtimes \partial B_1^+$ cannot be mapped by $\rho_{K_{12}}$ to $O \rtimes \partial B_2^-$ if $n \geq 5$, which contradicts (1.4). Therefore Σ_4 cannot be an analytic extension of Σ_1 if $n \geq 5$. If $n = 3, 4$, however, (1.4) follows from (1.2) and therefore Σ_4 is an embedded analytic extension of Σ_1 , as claimed.

From here on, assume $n = 4$. Note that Σ_4 meets ∂Q^4 orthogonally. Therefore repeated reflections of \mathbb{R}^n across the hyperplanes $\{x_i = 2k+1\}$ for all $i = 1, 2, 3, 4$ and for all integers

k give rise to the desired complete embedded analytic minimal hypersurface Σ_P in \mathbb{R}^4 . Obviously Σ_P is periodic in each direction of the four coordinate axes of \mathbb{R}^4 .

Interestingly, Σ_4 can be interpreted as an equator in Q^4 between the two poles $p^+ = (1, 1, 1, 1)$ and $p^- = (-1, -1, -1, -1)$ of ∂Q^4 . Define two 4-prong polar grids $\gamma^+ = \cup_{i=1}^4 \ell_i^+$ containing p^+ and $\gamma^- = \cup_{i=1}^4 \ell_i^-$ containing p^- by

$$\ell_1^+ = \{(x_1, 1, 1, 1) : -1 \leq x_1 \leq 1\}, \dots, \ell_4^+ = \{(1, 1, 1, x_4) : -1 \leq x_4 \leq 1\},$$

$$\ell_1^- = \{(x_1, -1, -1, -1) : -1 \leq x_1 \leq 1\}, \dots, \ell_4^- = \{(-1, -1, -1, x_4) : -1 \leq x_4 \leq 1\}.$$

Let γ_ε^+ be an ε -tubular neighborhood of γ^+ in Q^4 and γ_ε^- that of γ^- in Q^4 . Then the following four sets are diffeomorphic:

$$(1.6) \quad \partial\gamma_\varepsilon^+ \setminus \partial Q^4 \approx P_4 \approx \Sigma_4 \approx \partial\gamma_\varepsilon^- \setminus \partial Q^4.$$

It is in this sense that Σ_4 is called an equator between the two poles.

Define the 1-dimensional grid γ_∞^+ (γ_∞^- , respectively) in \mathbb{R}^4 to be the set of all lines parallel to the four coordinate axes, consisting of all the points (x_1, x_2, x_3, x_4) three components of which are integers $\equiv 1 \pmod{4}$ ($\equiv -1 \pmod{4}$, respectively). Then Σ_P can be viewed “roughly” as an equi-distance set of the grids γ_∞^+ and γ_∞^- in the following sense. Let

$$(2Q)^4 = [-3, 1]^4 = \{(x_1, x_2, x_3, x_4) : -3 \leq x_i \leq 1\}.$$

Identifying the two points on the parallel faces of $(2Q)^4$, one can make $(2Q)^4$ into a four-dimensional torus T^4 . With this identification $\Sigma_P \cap (2Q)^4$ becomes a compact 3-dimensional embedded minimal hypersurface Σ'_P in T^4 . It follows from (1.6) that Σ'_P is diffeomorphic to the boundary of a tubular neighborhood of γ_∞^+ in T^4 , and to that of γ_∞^- in T^4 as well. One can foliate $T^4 \setminus (\gamma_\infty^+ \cup \gamma_\infty^-)$ by a 1-parameter family of 3-dimensional hypersurfaces which are diffeomorphic to the boundary of a tubular neighborhood of γ_∞^+ and which sweep out T^4 from γ_∞^+ to γ_∞^- . Applying the minimax argument, one can find a compact embedded minimal hypersurface Σ_T from this family of hypersurfaces. Σ_T should be the same as Σ'_P . And one easily sees that $\pi_1(\Sigma'_P)$ is the free group with 4 generators. The hypersurface Σ_P divides \mathbb{R}^4 into two congruent labyrinths as one of them is mapped to the other by $\rho_{K_{12}}$.

In conclusion, we summarize the properties of Σ_P as follows.

Theorem 1.1. *There exists a minimal hypersurface Σ_P in \mathbb{R}^4 which generalizes the Schwarz P -surface of \mathbb{R}^3 with the following properties:*

- a) Σ_P is embedded and periodic in each direction of the four coordinate axes of \mathbb{R}^4 .
- b) Σ_P divides \mathbb{R}^4 into two congruent labyrinths.
- c) One can normalize the coordinates of \mathbb{R}^4 such that Σ_P has period 4 in each direction. Moreover, for every point $p \in \mathbb{R}^4$ with coordinates $(2k, 2l, 2m, 2n)$, k, l, m, n : integers, three mutually orthogonal planes pass through p and totally lie in Σ_P .
- d) Let T^4 be the 4-dimensional torus obtained by identifying the parallel faces of the cube $[-3, 1]^4$ in \mathbb{R}^4 . Then $\Sigma_P \cap [-3, 1]^4$ becomes a compact embedded minimal hypersurface Σ'_P in T^4 . Let $\gamma^4 \subset \mathbb{R}^2 \subset \mathbb{R}^4$ be a four-leaved rose, i.e., the union of four Jordan curves which intersect each other only at one given point. Then Σ'_P is diffeomorphic to the boundary of a tubular neighborhood of γ^4 in \mathbb{R}^4 and $\pi_1(\Sigma'_P)$ is the free group with 4 generators.

Remark 1. In conclusion, Schwarz’s minimal surface has been constructed in \mathbb{R}^3 and \mathbb{R}^4 but not in \mathbb{R}^n for $n \geq 5$. Strangely, this situation is similar to a famous classical problem in algebra: solvability of the cubic and quartic equations in radicals and unsolvability of the quintic. This may not be a pure coincidence, remarking that permutations of $\{x_1, \dots, x_n\}$ are critically used in the construction of Σ_4 and that Galois theory is based on the group of permutations. Moreover, as the roots of an algebraic equation are required to be expressed

only with the radicals, we have strongly required that the spine L_n be totally geodesic.

Remark 2. Our fundamental piece Σ_1 can be analytically extended to a complete embedded minimal hypersurface in \mathbb{R}^n for $n = 3, 4$, but not for $n \geq 5$. However, our guess is that such a complete embedded minimal hypersurface Σ_P should exist even in \mathbb{R}^n for $n \geq 5$. Near Σ_1 there should exist an analytic minimal hypersurface Σ'_1 whose boundary is more flexible than totally geodesic $\Sigma_1 \cap \Gamma_1$ and which is orthogonal to ∂Q^n so that Σ'_1 may extend to a complete embedded minimal hypersurface Σ_P in \mathbb{R}^n . Σ_P should be a minimax solution in a 1-parameter family of hypersurfaces sweeping out \mathbb{R}^n from γ_∞^+ to γ_∞^- . Here γ_∞^+ and γ_∞^- are the dual pair of all lines consisting of the points (x_1, \dots, x_n) , $(n-1)$ -components of which are integers $\equiv 1 \pmod{4}$ and $\equiv -1 \pmod{4}$, respectively.

2. SCHWARZ'S D -SURFACE

Schwarz's D -surface R is one of the simplest among dozens of triply periodic minimal surfaces in \mathbb{R}^3 . Its fundamental piece R_0 spans the skew quadrilateral with vertex angles $\pi/3, \pi/2, \pi/2, \pi/2$. It is interesting to notice that R_0 is a quarter of Schwarz's initial surface S_0 (Figure 8). Therefore Schwarz's P -surface and D -surface are the conjugate minimal surfaces.

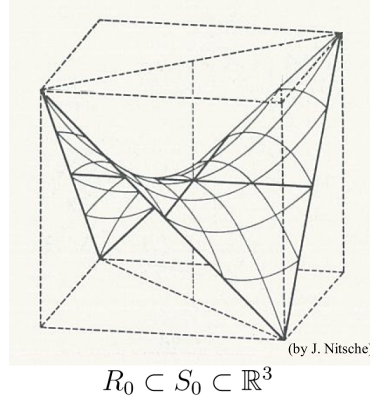


Figure 8

Thanks to the single vertex angle of $\pi/3$ in R_0 , six congruent pieces surrounding that vertex constitute a hexagonal minimal surface R_1 whose vertex angles are all $\pi/2$. (See Figure 9.)

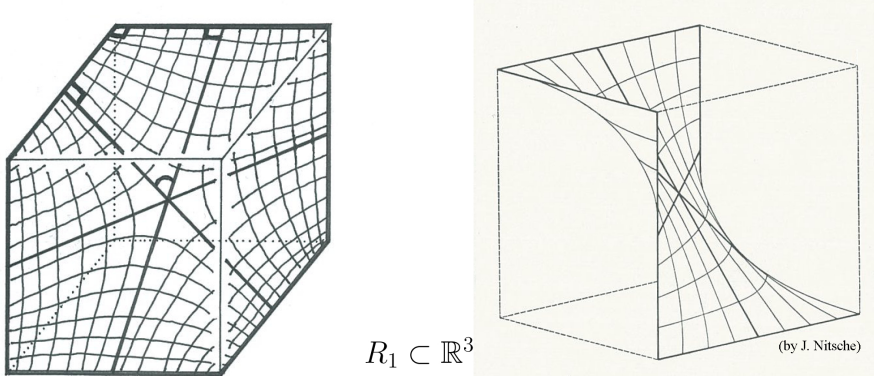


Figure 9

Since ∂R_1 is a subset of the 1-skeleton of a cube, R_1 can be extended to the complete embedded minimal surface R . We can generalize this nice property of R_1 in higher dimension to construct the higher-dimensional Schwarz D -surface in \mathbb{R}^n for any n as follows.

Theorem 2.1. *There exists an $(n-1)$ -dimensional Schwarz's D -surface Σ_D in \mathbb{R}^n for any $n \geq 4$:*

- a) Σ_D is complete and embedded.
- b) Σ_D is periodic in every direction of the coordinate axes of \mathbb{R}^n .
- c) If Σ_D is normalized to have period 2 in each coordinate direction, at every point $p \in \mathbb{R}^n$ with odd integer coordinates Σ_D completely contains $n-1$ $(n-2)$ -planes.

Proof. In the preceding section Σ_4 is interpreted as an equator in Q^4 between the two poles $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$. Here we introduce another type of equator in $\tilde{Q}^n := [0, 1]^n$ between the poles $p^0 = (0, \dots, 0)$ and $p^1 = (1, \dots, 1)$ in \tilde{Q}^n . \tilde{Q}^n has $2n$ faces $F_i^0 := \{x_i = 0\} \cap \partial\tilde{Q}^n$ and $F_i^1 := \{x_i = 1\} \cap \partial\tilde{Q}^n$ for $i = 1, \dots, n$. Define

$$F^0 = \bigcup_{i=1}^n F_i^0, \quad F^1 = \bigcup_{i=1}^n F_i^1, \quad \Gamma_2 = F^0 \cap F^1.$$

Clearly

$$\Gamma_2 = \partial F^0 = \partial F^1.$$

Γ_2 is homeomorphic to \mathbb{S}^{n-2} . Among 2^n vertices of \tilde{Q}^n , Γ_2 contains $2^n - 2$ of them, leaving out only p^0 and p^1 . As a CW-complex \tilde{Q}^n has the $(n-2)$ -skeleton which consists of $(n-2)$ -dimensional cubes. The total number of $(n-2)$ -dimensional cubes in the $(n-2)$ -skeleton of \tilde{Q}^n is $2n(n-1)$. Half of them contains either p^0 or p^1 . Hence Γ_2 contains $n(n-1)$ cubes.

Let Π_2 be the orthogonal projection of \mathbb{R}^n onto the hyperplane $\tilde{P}_n = \{x_1 + \dots + x_n = 0\}$. Then $\Pi_2(\Gamma_2)$ bounds a convex region $U := \Pi_2(\tilde{Q}^n) \subset \tilde{P}_n$. The vertices of U are the projections under Π_2 of all the vertices of \tilde{Q}^n except for p^0 and p^1 . Since Γ_2 is the graph of a piecewise linear function defined on $\Pi_2(\Gamma_2)$, Jenkins-Serrin's theorem gives a unique minimal hypersurface Σ_5 spanning Γ_2 as a graph over U . $\Sigma_5 = R_1$ in case $n = 3$. Obviously,

$$\Sigma_5 \subset \tilde{Q}^n \text{ because } \Gamma_2 \subset \partial\tilde{Q}^n.$$

As Γ_2 is invariant under the isometries of \mathbb{R}^n acting on $\{x_1, \dots, x_n\}$ as permutations, so is Σ_5 . Hence one can see that any pair of antipodal vertices $\{p, q\}$ (i.e., $\text{dist}(p, q) = \sqrt{n}$) of \tilde{Q}^n uniquely determines a minimal equator between them which is congruent to Σ_5 . Let's denote this minimal equator by $\Sigma_{\{p, q\}}$.

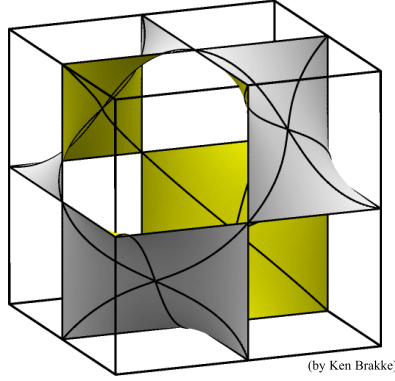
Define

$$2\tilde{Q}^n = [-1, 1] \times \dots \times [-1, 1] \subset \mathbb{R}^n.$$

The hyperplanes $\{x_i = 0\}, i = 1, \dots, n$, split $2\tilde{Q}^n$ into 2^n subcubes each of which is congruent to \tilde{Q}^n . One can make $2\tilde{Q}^n$ into an n -dimensional checkerboard by selecting the congruent subcubes in an alternating way. Let's denote the "black" part of the checkerboard containing \tilde{Q}^n by $\frac{2\tilde{Q}^n}{2}$. In each subcube of $\frac{2\tilde{Q}^n}{2}$ we want to put a minimal hypersurface congruent to Σ_5 as follows. Let Q be a copy of \tilde{Q}^n in $\frac{2\tilde{Q}^n}{2}$. Q has a unique vertex p_Q which is antipodal to O and then Q has a unique minimal equator $\Sigma_{\{O, p_Q\}}$ determined by the antipodal pair $\{O, p_Q\}$.

Combining all the minimal hypersurfaces $\Sigma_{\{O, p_Q\}}$ in each subcube Q of $\frac{2\tilde{Q}^n}{2}$, we define

$$\Sigma_6 = \bigcup_{Q \subset \frac{2\tilde{Q}^n}{2}} \Sigma_{\{O, p_Q\}}.$$



$$\Sigma_6 \subset \mathbb{R}^3$$

Figure 10

Since $\Gamma_2 = \partial F^0$ and $F^0 \subset \cup_i \{x_i = 0\}$, $\partial\Sigma_6$ is a subset of $\cup_i \{x_i = 0\}$. And since $\Gamma_2 = \partial F^1$ and $F^1 \subset \partial(2\tilde{Q}^n)$, $\partial\Sigma_6$ lies on the boundary of $2\tilde{Q}^n$. Therefore

$$(2.1) \quad \partial\Sigma_6 = \partial(2\tilde{Q}^n) \cap \bigcup_{i=1}^n \{x_i = 0\}.$$

Let q_i^+, q_i^- be the points on the x_i -axis whose x_i -coordinates equal 1, -1 , respectively. Then $q_i^+, q_i^- \in \Gamma_2$ and $\Gamma_2 \subset F_i^1 = \{x_i = 1\} \cap \partial\tilde{Q}^n$ in a neighborhood of q_i^+ for all $i = 1, \dots, n$. Hence Σ_5 is tangent to the face F_i^1 of \tilde{Q}^n at q_i^+ . It follows that Σ_6 is also tangent to the faces of $2\tilde{Q}^n$ at q_1^+, \dots, q_n^+ and at q_1^-, \dots, q_n^- .

In order to extend Σ_6 into a complete minimal hypersurface we need to understand the behavior of Σ_6 near the point $q_1^+ = (1, 0, \dots, 0) \in \Gamma_2$. In a neighborhood of q_1^+ Σ_5 is a graph over $V := \{(1, x_2, \dots, x_n) : x_i \geq 0, i = 2, \dots, n\} \subset \{x_1 = 1\}$. $\partial\Sigma_5$ contains all the $(n-2)$ -planes $\{x_1 = 1\} \cap \{x_i = 0\} \cap \partial V$ in a neighborhood of q_1^+ . Hence by the 180° -rotations of Σ_5 around all these $(n-2)$ -planes Σ_5 can be analytically extended to a minimal hypersurface Σ_7 which is a graph over $\{x_1 = 1\}$ in the same neighborhood. For $i = 2, \dots, n$, let ρ_i be the rotation of \mathbb{R}^n about the $(n-2)$ -plane $\{x_1 = 1\} \cap \{x_i = 0\}$ and let λ_i be the reflection in \mathbb{R}^n across the $(n-1)$ -plane $\{x_i = 0\}$. Since

$$\rho_2(x_1, \dots, x_n) = (2 - x_1, -x_2, x_3, \dots, x_n),$$

one gets

$$\rho_i \circ \rho_j = \lambda_i \circ \lambda_j$$

and hence

$$\rho_i \circ \rho_j \left(\frac{2\tilde{Q}^n}{2} \right) = \frac{2\tilde{Q}^n}{2}, \quad \rho_i \circ \rho_j(O) = O, \quad \rho_i \circ \rho_j(\Sigma_6) = \Sigma_6.$$

It follows that

$$(2.2) \quad \Sigma_6 = \Sigma_7 \text{ in a neighborhood of } q_1^+.$$

Remember that each subcube Q of $\frac{2\tilde{Q}^n}{2}$ in the checkerboard has a unique vertex p_Q antipodal to O and contains a unique minimal equator $\Sigma_{\{O, p_Q\}}$. Let $\mathcal{L} = \bigcup_{Q \subset \frac{2\tilde{Q}^n}{2}} \{p_Q\}$. \mathcal{L} forms an alternating subset in the set of 2^n vertices of $2\tilde{Q}^n$. Clearly \mathcal{L} consists of 2^{n-1} vertices and completely determines Σ_6 in the sense that $\Sigma_6 = \cup_{q \in \mathcal{L}} \Sigma_{\{O, q\}}$. Consider $\mathcal{L} \cap$

$\{x_1 = -1\}$ which consists of 2^{n-2} vertices of $2\tilde{Q}$. Choose any $q \in \mathcal{L} \cap \{x_1 = -1\}$ and let τ be the parallel translation of \mathbb{R}^n by 2 in the direction of x_1 -axis. Then

$$\tau(O) = (2, 0, \dots, 0), \quad \tau(q) \in \{x_1 = 1\}, \quad \tau(q) \notin \mathcal{L}.$$

However, there exists $\bar{q} \in \mathcal{L}$ such that $\tau(q) = \rho_i(\bar{q})$ for some $i = 2, \dots, n$. Moreover, $\tau(O) = \rho_i(O)$. Therefore we have

$$\tau(\Sigma_{\{O, q\}}) = \rho_i(\Sigma_{\{O, \bar{q}\}}).$$

Since $\Sigma_{\{O, \bar{q}\}} \subset \Sigma_6$ and $\rho_i(\Sigma_6) = \Sigma_7$ in a neighborhood of q_1^+ , it follows that

$$(2.3) \quad \tau(\Sigma_6) = \Sigma_7 \text{ in a neighborhood of } q_1^+.$$

Viewing Σ_7 as a minimal graph over $\{x_1 = 1\}$ in a neighborhood of q_1 , we see that the sign of Σ_7 is alternating on the components of $\{x_1 = 1\} \setminus \bigcup_{i=2}^n \{x_i = 0\}$. In a neighborhood of q_1^+ Σ_6 constitutes the part where Σ_7 is negative and $\tau(\Sigma_6)$ positive.

The point q_1^+ is the center of the face $\{x_1 = 1\}$ of $2\tilde{Q}^n$. Again by the invariance of Σ_5 under the permutations of $\{x_1, \dots, x_n\}$ the property of Σ_6 around q_1^+ as in (2.2) and (2.3) should also hold around the center q_i^+ of every face $\{x_i = 1\}$ of $2\tilde{Q}^n$. Hence we can extend Σ_6 into the complete embedded minimal hypersurface Σ_D by periodically translating Σ_6 (with period of 2) in every direction of the coordinate axes of \mathbb{R}^n :

$$\Sigma_D = \bigcup_{k_1, \dots, k_n: \text{integers}} \tau_{2k_1, \dots, 2k_n}(\Sigma_6),$$

where $\tau_{2k_1, \dots, 2k_n} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the parallel translation defined by

$$\tau_{2k_1, \dots, 2k_n}(x_1, \dots, x_n) = (x_1 + 2k_1, \dots, x_n + 2k_n).$$

By the removable singularity theorem [HL] Σ_D is analytic everywhere. Finally (2.1) implies that Σ_D contains $n - 1$ $(n - 2)$ -planes at q_i^\pm with odd integer coordinates. \square

3. SCHERK'S SECOND SURFACE

Scherk's minimal surfaces were found in 1834. After the catenoid(1744) and helicoid(1774), they were the third example(s) of minimal surfaces. Scherk used the method of separation of variables to find the equations

$$z = \log \cos x - \log \cos y \quad \text{and} \quad \sin z = \sinh x \cdot \sinh y$$

for the first surface and the second surface, respectively. Scherk's first surface is doubly periodic and the second surface singly periodic. It turns out that these two are conjugate minimal surfaces. The second surface is asymptotic to two orthogonal planes. In fact, H. Karcher [K] has found that there exist minimal saddle towers in \mathbb{R}^3 which are asymptotic to k planes intersecting each other along a line at equal angles of π/k for any integer $k \geq 2$. In this section we construct the higher dimensional generalizations of Scherk's second surface and the saddle towers. These hypersurfaces are asymptotic to k hyperplanes meeting each other at π/k -angles for any integer $k \geq 2$. The key idea of our method is to use the catenoid as a barrier in the Dirichlet problem. It should be mentioned that F. Pacard [P] has constructed similar hypersurfaces for $k = 2$ using a different method (desingularization procedure).

Theorem 3.1. *For any integer $k \geq 2$ there exists an embedded minimal hypersurface Σ_S in \mathbb{R}^n satisfying the following properties:*

a) Σ_S is asymptotic to k hyperplanes Π_1, \dots, Π_k meeting each other along the $(n-2)$ -plane $P^{n-2} := \{x_1 = 0, x_n = 0\}$ at equal angles of π/k .

b) Σ_S is periodic in $n - 2$ pairwise orthogonal directions of P^{n-2} .

c) Given any positive real numbers a_2, \dots, a_{n-1} , consider the union \mathcal{P}^{n-3} of all the $(n-3)$ -planes $\{x_1 = 0, x_n = 0, x_2 = ma_2\}, \{x_1 = 0, x_n = 0, x_3 = ma_3\}, \dots, \{x_1 = 0, x_n = 0, x_{n-1} = ma_{n-1}\}$ in P^{n-2} for every integer m . \mathcal{P}^{n-3} divides P^{n-2} into $(n-2)$ -dimensional rectangular cubes which are all congruent to $(0, a_2) \times (0, a_3) \times \dots \times (0, a_{n-1})$. Let ℓ_1, \dots, ℓ_k be the lines in the $x_1 x_n$ -plane which are contained in Π_1, \dots, Π_k , respectively, so that $\Pi_i = P^{n-2} \times \ell_i$, $i = 1, \dots, k$. Then Σ_S contains $\mathcal{P}^{n-3} \times \ell_i$ for all $i = 1, \dots, k$.

To prove this theorem we need to introduce the higher-dimensional catenoid $\mathcal{C}^{n-1} \subset \mathbb{R}^n$. \mathcal{C}^{n-1} is obtained by rotating a generating curve $C : x_n = f(x_1)$ of the $x_1 x_n$ -plane through the $SO(n-1)$ action on the $x_2 \dots x_n$ -plane. The resulting hypersurface has zero mean curvature if and only if

$$x_n x_n'' - (n-2)\{1 + (x_n')^2\} = 0.$$

It is interesting to note that \mathcal{C}^{n-1} lies in a slab of \mathbb{R}^n if $n \geq 4$. Since the minimality of a hypersurface is invariant under homothety, we can assume that \mathcal{C}^{n-1} lies in the slab $\{-1 < x_1 < 1\}$ and is asymptotic to the boundaries of the slab. Let's define the upper half catenoid

$$\frac{1}{2}\mathcal{C}^{n-1} = \mathcal{C}^{n-1} \cap \{x_n \geq 0\}.$$

$\frac{1}{2}\mathcal{C}^{n-1}$ is the graph of a nonnegative function $x_n = g(x_1, \dots, x_{n-1})$. Since one can find $a > 0$ such that $f(x_1) \geq a$ for $-1 < x_1 < 1$ and $f(0) = a$, the domain of definition of g contains the solid cylinder $D^{n-1} := \{-1 < x_1 < 1, x_2^2 + \dots + x_{n-1}^2 < a^2, x_n = 0\}$. We are going to use $\frac{1}{2}\mathcal{C}^{n-1}$ as a barrier in the proof of the theorem.

Proof. By the invariance of minimality of Σ_S under homothety we may assume

$$(3.1) \quad a_i < \frac{a}{\sqrt{n-2}} \quad \text{for all } i = 2, \dots, n-1.$$

Let $Q_b^{n-1} = [-b, b] \times [0, a_2] \times \dots \times [0, a_{n-1}]$ be a closed cube in the horizontal hyperplane $\{x_n = 0\}$. Then by (3.1) we have $Q_b^{n-1} \cap \{-1 < x_1 < 1\} \subset D^{n-1}$. For any integer $k \geq 2$, define a function on the infinite cube Q_∞^{n-1}

$$(3.2) \quad h_k(x_1, \dots, x_{n-1}) = c_k |x_1|,$$

where $c_k > 0$ is to be determined.

The graph of $x_n = h_k(x_1, \dots, x_{n-1})$ over Q_b^{n-1} is piecewise planar (V-shaped) with angle θ_k along the sharp edge over $\{0\} \times [0, a_2] \times \dots \times [0, a_{n-1}]$. Determine c_k in such a way that $\theta_k = \pi/k$. We want to replace $\text{graph}(h_k)$ with a minimal hypersurface by finding a function $\tilde{h}_{k,b}$ on Q_b^{n-1} such that $\tilde{h}_{k,b} = h_k$ on ∂Q_b^{n-1} and the graph of $x_n = \tilde{h}_{k,b}(x_1, \dots, x_{n-1})$ is minimal. By Jenkins-Serrin [JS] such a $\tilde{h}_{k,b}$ exists. From (3.2) we see that $h_k \leq c_k$ on Q_1^{n-1} . Hence

$$(3.3) \quad \tilde{h}_{k,1} \leq c_k \quad \text{on } Q_1^{n-1}.$$

Clearly we have

$$\tilde{h}_{k,b_1} < \tilde{h}_{k,b_2} \quad \text{on } Q_{b_1}^{n-1} \quad \text{if } b_1 < b_2.$$

When b increases, we need to show that $\tilde{h}_{k,b}$ cannot become much bigger than g on Q_1^{n-1} . Suppose $\tilde{h}_{k,1} = g + c_k$ at a point p_1 of Q_1^{n-1} . Since $g > 0$ on Q_1^{n-1} , from (3.3) we see that p_1 cannot be a boundary point of Q_∞^{n-1} . p_1 cannot be a boundary point of the slab $\{-1 < x_1 < 1\}$ either, because $g = \infty$ there. Hence p_1 must be an interior point of Q_1^{n-1} .

Then there should exist an interior point p_2 of Q_1^{n-1} and $c > 0$ such that $\tilde{h}_{k,1} \leq g + c_k + c$ on Q_1^{n-1} and equality holds at p_2 . But this contradicts the maximum principle. Hence

$$\tilde{h}_{k,b} < g + c_k \text{ on } Q_1^{n-1} \text{ for any } b \geq 1.$$

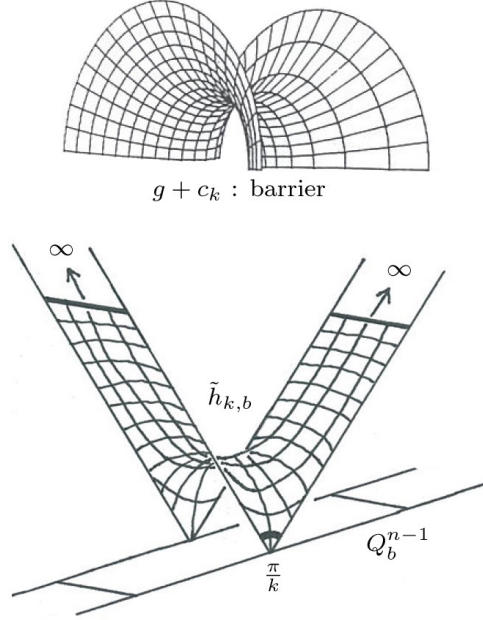


Figure 11

Therefore the limit \tilde{h}_k of $\tilde{h}_{k,b}$ as $b \rightarrow \infty$ exists on Q_1^{n-1} (see Figure 11) and

$$\tilde{h}_k \leq g + c_k \text{ on } Q_1^{n-1}.$$

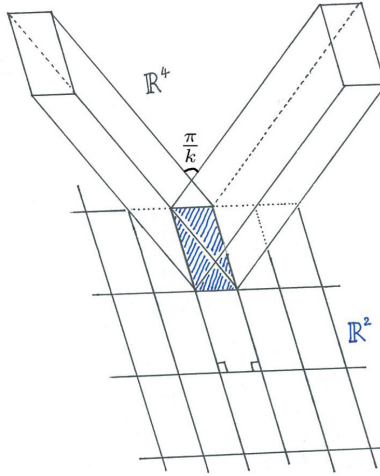


Figure 12

We claim that \tilde{h}_k exists on the infinite cube Q_∞^{n-1} as well. Note that

$$g \leq a \text{ on } \{0\} \times [0, a_2] \times \cdots \times [0, a_{n-1}].$$

Hence

$$\tilde{h}_{k,b}(x_1, \dots, x_{n-1}) \leq (a + c_k) + c_k|x_1|$$

on the boundaries of $[0, b] \times [0, a_2] \times \dots \times [0, a_{n-1}]$ and $[-b, 0] \times [0, a_2] \times \dots \times [0, a_{n-1}]$ for any $b > 0$. Thus

$$\tilde{h}_{k,b}(x_1, \dots, x_{n-1}) \leq (a + c_k) + c_k|x_1| \quad \text{on } Q_b^{n-1}$$

for any b and so \tilde{h}_k exists on Q_∞^{n-1} , as claimed. Clearly \tilde{h}_k is analytic.

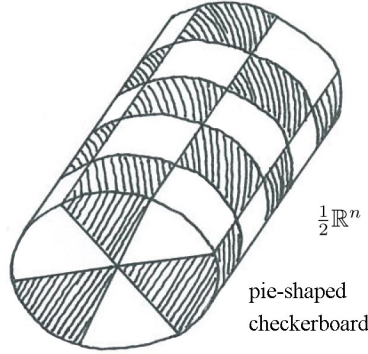


Figure 13

Let Σ_8 be the graph of $x_n = \tilde{h}_k(x_1, \dots, x_{n-1})$ on Q_∞^{n-1} . Σ_8 inherits all the symmetries of Q_∞^{n-1} , that is, Σ_8 is symmetric with respect to the $n-1$ vertical pairwise orthogonal hyperplanes of \mathbb{R}^n which divide each interval of Q_b^{n-1} into halves. Given an $(n-2)$ -dimensional rectangular cube Q^{n-2} , let's call $Q^{n-2} \times \mathbb{R}^2$ an $(n-2)$ -slab in \mathbb{R}^n . So a slab of \mathbb{R}^3 is called a 1-slab. The graph of the piecewise-linear function $x_n = h_k(x_1, \dots, x_{n-1})$ divides the $(n-2)$ -slab $[0, a_2] \times \dots \times [0, a_{n-1}] \times (x_1 x_n\text{-plane})$ into two components. The smaller one is (infinite) pie-shaped; let's denote it as V . As the two planar boundaries of V in the interior of the $(n-2)$ -slab make an angle of π/k , the $(n-2)$ -slab $[0, a_2] \times \dots \times [0, a_{n-1}] \times (x_1 x_n\text{-plane})$ can be divided into $2k$ pie-shaped domains congruent to V . \mathbb{R}^n can be divided into a tessellation \mathcal{T}_0 by $(n-2)$ -slabs which are all congruent to $[0, a_2] \times \dots \times [0, a_{n-1}] \times (x_1 x_n\text{-plane})$ and one of which is $[0, a_2] \times \dots \times [0, a_{n-1}] \times (x_1 x_n\text{-plane})$ itself. One can refine \mathcal{T}_0 into another tessellation \mathcal{T}_1 by dividing each $(n-2)$ -slab of \mathcal{T}_0 into $2k$ pie-shaped domains congruent to V . Let $\frac{1}{2}\mathbb{R}^n$ denote the union of all the pie-shaped domains in \mathcal{T}_1 which are chosen alternately such that $V \subset \frac{1}{2}\mathbb{R}^n$ (see Figure 13). $\frac{1}{2}\mathbb{R}^n$ is called the *pie-shaped checkerboard*.

Σ_8 is an embedded minimal hypersurface in V so that $\partial\Sigma_8$ is a subset of the $(n-2)$ -skeleton of V . Each pie-shaped domain V_0 of $\frac{1}{2}\mathbb{R}^n$ contains a unique minimal hypersurface Σ_{V_0} which is congruent to Σ_8 and whose boundary is a subset of the $(n-2)$ -skeleton of V_0 . Define

$$\Sigma_S = \bigcup_{V_0 \subset \frac{1}{2}\mathbb{R}^n} \Sigma_{V_0}.$$

Let V_1, V_2 be two neighboring pie-shaped domains of $\frac{1}{2}\mathbb{R}^n$ which share a nonempty subset K of their $(n-2)$ -skeletons. Then Σ_{V_1} is the 180° -rotation of Σ_{V_2} around K because of the symmetries of Σ_8 . Therefore Σ_S is a complete embedded analytic minimal hypersurface as described by a), b), c). \square

Remark 3. The catenoid can be used as a barrier to construct even Scherk's second surface and the saddle towers in \mathbb{R}^3 without appealing to the Weierstrass representation formula. Moreover, H. Karcher's helicoidal saddle towers [K] can be constructed in this

way: Deform V into V_0 which is invariant under a screw motion rotating around the x_2 -axis and tessellate \mathbb{R}^3 by the domains congruent to V_0 ; use the half catenoid as a barrier to construct a minimal surface Σ_{V_0} whose boundary is a subset of the 1-skeleton of V_0 ; keep rotating Σ_{V_0} around its boundaries by 180 degrees.

Remark 4. Higher dimensional Scherk's first surface Σ_{S_1} can be also constructed in \mathbb{R}^n by solving the Dirichlet problem on the domain $O \rtimes F$, where F is a face of the cube $[-1, 1]^{n-1} \subset \mathbb{R}^{n-1}$. But Σ_{S_1} has a self-intersection in case $n \geq 4$ because the tessellation of \mathbb{R}^{n-1} by the domains congruent to $O \rtimes F$ cannot generate the pyramid-shaped checkerboard $\frac{1}{2}\mathbb{R}^{n-1}$.

REFERENCES

- [B] D. E. Blair, *On a generalization of the catenoid*, Can. J. Math. **27** (1975), 231–236.
- [C] J. Choe, *On the existence of higher dimensional Enneper's surface*, Comment. Math. Helv. **71** (1996), 556–569.
- [CH] J. Choe and J. Hoppe, *Higher dimensional minimal submanifolds generalizing the catenoid and helicoid*, Tohoku Math. J. **65** (2013), 43–55.
- [HL] R. Harvey and H.B. Lawson, *Extending minimal varieties*, Inv. Math. **28** (1975), 209–226.
- [JS] H. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math. **229** (1968), 170–187.
- [K] H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math. **62** (1988) pp. 83–114.
- [P] F. Pacard, *Higher dimensional Scherk's hypersurfaces*, J. Math. Pures Appl. **81** (2002), 241–258.
- [S] H.A. Schwarz, *Gesammelte Mathematische Abhandlungen*, Band I und II. Springer, Berlin 1890.

KOREA INSTITUTE FOR ADVANCED STUDY, SEOUL, 02455, KOREA

E-mail address: `choe@kias.re.kr`

KTH, 100 44 STOCKHOLM, SWEDEN,

E-mail address: `hoppe@kth.se`